

Irreducibility of A-hypergeometric systems

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Abstract

We give an elementary proof of the Gel'fand-Kapranov-Zelevinsky theorem that non-resonant A-hypergeometric systems are irreducible. We also provide a proof of a converse statement

1 Introduction

Let $A \subset \mathbb{Z}^r$ be a finite set such that

1. The \mathbb{Z} -span of A is \mathbb{Z}^r .
2. There exists a linear form h such that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in A$.

Let $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{C}^r$. At the end of the 1980's Gel'fand, Kapranov and Zelevinsky [4], [5], [6] developed a theory of hypergeometric functions and equations which uses A and α as starting data. It turns out that the resulting equations contain the classical cases of Appell, Horn, Lauricella and Aomoto hypergeometric functions.

Denote $A = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ (with $N > r$). Writing the vectors \mathbf{a}_i in column form we get the so-called A-matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rN} \end{pmatrix}$$

For $i = 1, 2, \dots, r$ consider the first order differential operators

$$Z_i = a_{i1}v_1\partial_1 + a_{i2}v_2\partial_2 + \cdots + a_{iN}v_N\partial_N$$

where $\partial_j = \frac{\partial}{\partial v_j}$ for all j .

Let

$$L = \{(l_1, \dots, l_N) \in \mathbb{Z}^N \mid l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + \cdots + l_N\mathbf{a}_N = \mathbf{0}\}$$

be the lattice of integer relations between the elements of A . For every $\mathbf{l} \in L$ we define the so-called box-operator

$$\square_{\mathbf{l}} = \prod_{l_i > 0} \partial_i^{l_i} - \prod_{l_i < 0} \partial_i^{-l_i}$$

The system of differential equations

$$\begin{aligned} (Z_i - \alpha_i)\Phi &= 0 & (i = 1, \dots, r) \\ \square_{\mathbf{l}}\Phi &= 0 & \mathbf{l} \in L \end{aligned}$$

is known as the system of *A-hypergeometric differential equations* and we denote it by $H_A(\alpha)$. We like to remark that independently, and at around the same time, B.Dwork arrived at a similar setup for generalised hypergeometric functions. The system of A-hypergeometric equations is implicit in his book [3].

Let $K = \mathbb{C}(v_1, \dots, v_N)$ and let $\mathcal{H}_A(\alpha)$ be the left ideal in $K[\partial_1, \dots, \partial_N]$ generated by the operators from $H_A(\alpha)$. The quotient $K[\partial_1, \dots, \partial_N]/\mathcal{H}_A(\alpha)$ is a K -module. Its K -rank is called the rank of the system $H_A(\alpha)$. Furthermore, the system is called *non-resonant* if the set $\alpha + \mathbb{Z}^r$ has empty intersection with the boundary of $C(A)$. The system is called *resonant* if the intersection is non-empty.

In [6], (corrected in [8]) and [1, Corollary 5.20] the following theorem is shown.

Theorem 1.1 (GKZ, Adolphson) *Suppose either one of the following conditions holds,*

1. *the toric ideal I_A in $\mathbb{C}[\partial_1, \dots, \partial_N]$ generated by the box operators has the Cohen-Macaulay property.*
2. *The system $H_A(\alpha)$ is non-resonant.*

Then the rank of $H_A(\alpha)$ is finite and equals the volume of the convex hull $Q(A)$ of the points of A . The volume is normalized so that a minimal $(r-1)$ -simplex with integer vertices in $h(\mathbf{x}) = 1$ has volume 1.

Let p be a generic point in $(\mathbb{C}^*)^N$ (the space with coordinates v_1, \dots, v_N). Then it is known that the dimension of the \mathbb{C} -vector space of local power series solutions around p of $H_A(\alpha)$ equals the rank of $H_A(\alpha)$.

The K -module $K[\partial_1, \dots, \partial_N]/\mathcal{H}_A(\alpha)$ has a natural left action by the operators ∂_i , so it is a D -module. We shall say that the system $H_A(\alpha)$ is *irreducible* if this D -module has no submodules beside 0 and the module itself. We call it *reducible* otherwise. Gel'fand, Kapranov and Zelevinsky proved in [7, Thm 2.11] the following beautiful theorem.

Theorem 1.2 (GKZ, 1990) *Suppose the system $H_A(\alpha)$ is non-resonant. Then $H_A(\alpha)$ is irreducible.*

The proof uses the theory of perverse sheaves and is hard to follow for someone without this background. It is the purpose of the present paper to give a more elementary proof of this theorem. This is done in Section 6. In addition we prove a converse statement, namely the following.

Theorem 1.3 *Suppose that the toric ideal I_A has the Cohen-Macaulay property and suppose that the convex hull $Q(A)$ is not a pyramid. If the system $H_A(\alpha)$ is resonant, then it is reducible.*

As far as we could see the latter theorem is not stated as such in the papers of Gel'fand, Kapranov and Zelevinsky or any other papers. The condition that $Q(A)$ is not a pyramid means that we like to avoid the situation where A contains $N-1$ points in an $r-2$ -dimensional affine plane and only one outside of it. It is not hard to see that $Q(A)$ is a pyramid if and only if for every index $i \in \{1, \dots, N\}$ there exists $\mathbf{l} \in L$ such that $l_i \neq 0$. Suppose $Q(A)$ is a pyramid with top \mathbf{a}_1 . Then one easily sees that the box-operators do not contain ∂_1 . Hence there exists $\beta \in \mathbb{R}$ such that the solutions of $H_A(\alpha)$ have the form $v_1^\beta F(v_2, \dots, v_N)$. In case $\beta = 0$, so all solutions independent of v_1 , the vector of parameters lies in the bottom of the pyramid, which is the affine space spanned by $\mathbf{a}_2, \dots, \mathbf{a}_N$.

2 Contiguity

Consider the system $H_A(\alpha)$,

$$\square_1 \Phi = 0, \quad \mathbf{l} \in L, \quad Z_j \Phi = \alpha_j \Phi, \quad j = 1, \dots, r.$$

Apply the operator ∂_i from the left. We obtain,

$$\square_1 \partial_i \Phi = 0, \quad \mathbf{l} \in L, \quad Z_j \partial_i \Phi = -a_{ji} \partial_i \Phi, \quad j = 1, \dots, r.$$

In other words, $F \mapsto \partial_i F$ maps the solution space of $H_A(\alpha)$ to the solution space of $H_A(\alpha - \mathbf{a}_i)$.

We can phrase this alternatively in terms of D-modules. Denote by $\mathcal{H}_A(\alpha)$ the left ideal in $K[\partial]$ generated by the hypergeometric operators \square_1 and Z_j . Then the map $P \mapsto P\partial_i$ gives a D-module homomorphism $K[\partial]/\mathcal{H}_A(\alpha - \mathbf{a}_i) \rightarrow K[\partial]/\mathcal{H}_A(\alpha)$. We are interested in the cases when this is a D-module isomorphism or, equivalently, whether $F \mapsto \partial_i F$ gives an isomorphism of solution spaces.

The following Theorem was first proven by B.Dwork in his book [3, Thm 6.9.1]. Another proof was given in [2, Lemma 7.10]. We present an adaptation of Dwork's ideas into a language which is quite different from Dwork's.

Theorem 2.1 (Dwork) *Suppose $H_A(\alpha)$ is non-resonant. Then the map $F \mapsto \partial_i F$ yields an isomorphism between the solution spaces of $H_A(\alpha)$ and $H_A(\alpha - \mathbf{a}_i)$.*

For the proof we need an extra Lemma and some notation. Suppose the positive cone $C(A)$ is given by a finite set \mathcal{F} of linear inequalities $l(\mathbf{x}) \geq 0$, $l \in \mathcal{F}$. Assume moreover that the linear forms l are integral valued on \mathbb{Z}^r and normalise them so that the greatest common divisor of all values is 1.

Consider the integral points in $C(A)$. It is not necessarily true that every point in $C(A) \cap \mathbb{Z}^r$ is a linear combination of the \mathbf{a}_i with non-negative integer coefficients. However, we do have the following Lemma.

Lemma 2.2 *There exists a point $\mathbf{p} \in C(A) \cap \mathbb{Z}^r$ such that $(\mathbf{p} + C(A)) \cap \mathbb{Z}^r \subset \mathbb{Z}_{\geq 0}A$ where $\mathbb{Z}_{\geq 0}A$ is the span of A with non-negative integer coefficients.*

Proof It is clear that there exists a positive integer δ such that for any point $(\lambda_1, \dots, \lambda_N) \in L \otimes \mathbb{R}$ there exists $(m_1, \dots, m_N) \in L$ such that $|m_i - \lambda_i| \leq \delta$. Let us take $\mathbf{p} = \delta(\mathbf{a}_1 + \dots + \mathbf{a}_N)$.

Suppose we are given a point $\mathbf{n} \in (\mathbf{p} + C(A)) \cap \mathbb{Z}^r$. Then there exist $\lambda_i \in \mathbb{R}_{\geq \delta}$ and integers n_1, \dots, n_N such that $\mathbf{n} = \lambda_1 \mathbf{a}_1 + \dots + \lambda_N \mathbf{a}_N = n_1 \mathbf{a}_1 + \dots + n_N \mathbf{a}_N$. The point $(\lambda_1 - n_1, \dots, \lambda_N - n_N)$ lies in $L \otimes \mathbb{R}$. Hence there exists $(m_1, \dots, m_N) \in L$ such that $|\lambda_i - n_i - m_i| \leq \delta$ for $i = 1, \dots, N$. Since $\lambda_i \geq \delta$ for every i we find that $n_i + m_i \geq 0$. Hence $\mathbf{n} = n_1 \mathbf{a}_1 + \dots + n_N \mathbf{a}_N = (n_1 + m_1) \mathbf{a}_1 + \dots + (n_N + m_N) \mathbf{a}_N$, hence $\mathbf{n} \in \mathbb{Z}_{\geq 0}A$. \square

Proof of Thm 2.1. We will construct an operator $P \in K[\partial]$ such that $P\partial_i \equiv 1 \pmod{\mathcal{H}_A(\alpha)}$. In particular, $F \mapsto P(F)$ would be the inverse of ∂_i , which establishes the isomorphism.

For any $l \in \mathcal{F}$ and any differential operator $\partial^{\mathbf{u}} = \partial_1^{u_1} \dots \partial_N^{u_N}$ we define the valuation $val_l(\partial^{\mathbf{u}}) = \sum_{j=1}^N u_j l(\mathbf{a}_j)$. More generally, for any differential operator $P \in K[\partial]$ we define $val_l(P)$ to be the minimal valuation of all terms in P .

Let \mathbf{p} be as in Lemma 2.2. Suppose $val_l(\partial^{\mathbf{u}}) \leq val_l(\partial^{\mathbf{w}}) + l(\mathbf{p})$ for every $l \in \mathcal{F}$. Hence $\sum_{j=1}^N l((w_j - u_j) \mathbf{a}_j) \geq l(\mathbf{p})$ for all $l \in \mathcal{F}$. So, according to Lemma 2.2 $\sum_{j=1}^N (w_j - u_j) \mathbf{a}_j$ is a lattice point in $\mathbb{Z}_{\geq 0}A$. Hence there exist non-negative integers w'_j such that $\sum_{j=1}^N w'_j \mathbf{a}_j = \sum_{j=1}^N (w_j - u_j) \mathbf{a}_j$. Hence $\partial^{\mathbf{w}}$ is equivalent modulo the box operator $\square_{\mathbf{w}-\mathbf{w}'-\mathbf{u}}$ with $\partial^{\mathbf{w}'} \partial^{\mathbf{u}}$.

Let $l \in \mathcal{F}$ be given. We show that modulo the ideal $\mathcal{H}_A(\alpha)$, the operator $\partial^{\mathbf{u}}$ is equivalent to an operator P such that $val_l(P) > val_l(\partial^{\mathbf{u}})$ and $v_{l'}(P) \geq v_{l'}(\partial^{\mathbf{u}})$ for all $l' \in \mathcal{F}$, $l' \neq l$. Let $Z_l = -l(\alpha) + \sum_{j=1}^N l(\mathbf{a}_j) v_j \partial_j$. Notice that $Z_l \in \mathcal{H}_A(\alpha)$ and $\partial^{\mathbf{u}} Z_l = Z_l \partial^{\mathbf{u}} + l(\mathbf{u}) \partial^{\mathbf{u}}$. Hence,

$$\sum_{j=1}^N l(\mathbf{a}_j) v_j \partial_j \partial^{\mathbf{u}} \equiv l(\alpha - \mathbf{u}) \partial^{\mathbf{u}} \pmod{\mathcal{H}_A(\alpha)}.$$

For each term on the left we have $l(\mathbf{a}_j) \neq 0 \Rightarrow val_l(\partial_j \partial^{\mathbf{u}}) > val_l(\partial^{\mathbf{u}})$. Since, by non-resonance, $l(\alpha - \mathbf{u}) \neq 0$ our assertion is proven. Choose $k_l \in \mathbb{Z}_{\geq 0}$ for every $l \in \mathcal{F}$. By repeated application of our principle we see that any monomial $\partial^{\mathbf{u}}$

is equivalent modulo $\mathcal{H}_A(\alpha)$ to an operator P with $\text{val}_l(P) \geq k_l + \text{val}_l(\partial^{\mathbf{u}})$ for all $l \in \mathcal{F}$.

In particular, there exists an operator P , equivalent to 1 and $\text{val}_l(P) \geq \text{val}_l(\partial_i) + l(\mathbf{p})$ for every $l \in \mathcal{F}$. Then, P is equivalent to an operator $P'\partial_i$. Summarising, $1 \equiv P'\partial_i \pmod{\mathcal{H}_A(\alpha)}$. So $F \mapsto \partial_i F$ is injective on the solution space of $H_A(\alpha)$. \square

There is another instance when $F \mapsto \partial_i F$ is an isomorphism of solution spaces.

Theorem 2.3 *Suppose that the toric ideal I_A has the Cohen-Macaulay property, that $Q(A)$ is not a pyramid and that $H_A(\alpha)$ is an irreducible system. Then $F \mapsto \partial_i F$ gives an isomorphism of solution spaces of $H_A(\alpha)$ and $H_A(\alpha - \mathbf{a}_i)$.*

Proof. Since $H_A(\alpha)$ is irreducible, the kernel of $F \mapsto \partial_i F$ is either trivial or the entire solution space. In the first case we are done, the map is injective and the solution spaces have the same dimension (because I_A has the Cohen-Macaulay property).

Now suppose we are in the second case, when $\partial_i F \equiv 0$ for every solution F of $H_A(\alpha)$. This is equivalent to the statement $\partial_i \in \mathcal{H}_A(\alpha)$. Let us write

$$\partial_i = \sum_{\lambda} A_{\lambda} \square_{\lambda} + \sum_{j=1}^r B_j (Z_j - \alpha_j).$$

The summation over the $\lambda \in L$ is supposed to be a finite summation. Let us assume that we have chosen the A_{λ} and B_i such that the maximum of the orders of the B_i is minimal. Call this minimum m . We assert that $m = 0$. Suppose $m > 0$.

We now work over the polynomial ring $R = \mathbb{C}(\mathbf{v})[X_1, \dots, X_N]$. For any differential operator P we write $P(\mathbf{X})$ for the polynomial we get after we replace ∂_j by X_j for all j in P . Write I_A for the ideal in R generated by the $\square_1(\mathbf{X})$. Since the quotient ring R/I_A is a Cohen-Macaulay ring, the linear forms $Z_i(\mathbf{X})$ form a regular sequence. In particular this means that if $P_1 Z_1(\mathbf{X}) + \dots + P_r Z_r(\mathbf{X}) = 0$ in R/I_A , then there exist polynomials η_{ij} with $\eta_{ij} = -\eta_{ji}$ such that $P_i = \sum_{j=1}^r \eta_{ij} Z_j(\mathbf{X})$ for $i = 1, \dots, r$.

Let us return to the A_{λ} and B_j above. Note that $(A_{\lambda} \square_{\lambda})(\mathbf{X}) = A_{\lambda}(\mathbf{X}) \square_{\lambda}(\mathbf{X})$ since the box-operators have constant coefficients. Denote the order m part of each B_j by $B_j^{(m)}$. Then the $m+1$ -st degree part of $\sum_j (B_j(Z_j - \alpha_j))(\mathbf{X})$ reads $\sum_j B_j^{(m)}(\mathbf{X}) Z_j(\mathbf{X})$. Since $m+1 > 1$ this degree $m+1$ part is zero in R/I_A . Hence there exist polynomials η_{jk} with $\eta_{jk} = -\eta_{kj}$ such that $B_j^{(m)}(\mathbf{X}) = \sum_{k=1}^r \eta_{jk} Z_k(\mathbf{X})$ in R/I_A . Denote by E_{jk} the differential operator which we get after we replace the variables X_b in η_{jk} by their counterparts ∂_b . Define $\tilde{B}_j = B_j - \sum_{k=1}^r E_{jk}(Z_k - \alpha_k)$ and note that \tilde{B}_j has order $< m$. Moreover,

$$\sum_{j=1}^r B_j(Z_j - \alpha_j) = \sum_{j=1}^r \tilde{B}_j(Z_j - \alpha_j) + \sum_{j,k=1}^r E_{jk}(Z_j - \alpha_j)(Z_k - \alpha_k).$$

The last sum, by virtue of the antisymmetry of the E_{jk} and the fact that $Z_j - \alpha_j$ and $Z_k - \alpha_k$ commute for all j, k , is equal to zero in R/I_A . Hence

$$\partial_i \equiv \sum_{j=1}^r \tilde{B}_j(Z_j - \alpha_j) \pmod{I_A}$$

where the \tilde{B}_j have order $< m$. This contradicts the minimality of m . Therefore we conclude that $m = 0$. In other words there exist $b_i \in \mathbb{C}(\mathbf{v})$ such that $\partial_i \equiv \sum_{j=1}^r b_j (Z_j - \alpha_j) \pmod{I_A}$. Since the box-operators all have order ≥ 2 this relation holds exact. It follows that there exist $\beta_j \in \mathbb{C}$ such that $v_i \partial_i =$

$\sum_{j=1}^r \beta_j (Z_j - \alpha_j)$. In other words there exists a linear form m on \mathbb{R}^r such that $m(\mathbf{a}_j) = 0$ for all $j \neq i$ and $m(\mathbf{a}_i) = 1$. But this implies that $Q(A)$ is a pyramid with \mathbf{a}_i as a top. \square

3 Resonant systems

In this section we prove Theorem 1.3. Suppose that $H_A(\alpha)$ is resonant and irreducible. Then, by Theorem 2.3 for any i the map $F \mapsto \partial_i F$ is an isomorphism of solution spaces of $H_A(\alpha)$ and $H_A(\alpha - \mathbf{a}_i)$. So we see that $H_A(\beta)$ is irreducible for any $\beta \in \mathbb{R}^r$ with $\beta \equiv \alpha \pmod{\mathbb{Z}^r}$. Since the system is resonant there exists such a β in a face F of $C(A)$. Suppose $A \cap F = \{\mathbf{a}_1, \dots, \mathbf{a}_t\}$. We assert that there exist non-trivial solutions of the form $f = f(v_1, \dots, v_t)$. Suppose that $s = \text{rank}(\mathbf{a}_1, \dots, \mathbf{a}_t)$. By an $SL(r, \mathbb{Z})$ change of coordinates we can see to it that F is given by $x_{s+1} = \dots = x_r = 0$. Then the coordinate a_{rj} of \mathbf{a}_j is zero for $i = s+1, \dots, r$ and $j = 1, \dots, t$. Also, $\beta_{s+1} = \dots = \beta_r = 0$. A solution $f = f(v_1, \dots, v_t)$ satisfies the homogeneity equations

$$\left(-\beta_i + \sum_{j=1}^t a_{ij} v_j \partial_j \right) f = 0, \quad i = 1, \dots, s.$$

Notice that the homogeneity equation with $i = s+1, \dots, r$ are trivial.

Consider the box-operator \square_λ with $\lambda \in L$. Write $\lambda = (\lambda_1, \dots, \lambda_N)$. The positive support is the set of indices i where $\lambda_i > 0$, the negative support is the set of indices i where $\lambda_i < 0$.

Suppose the positive support is contained in $1, 2, \dots, t$. Then $\sum_{\lambda_i > 0} \lambda_i \mathbf{a}_i$ is in \mathcal{F} . Hence $-\sum_{\lambda_i < 0} \lambda_i \mathbf{a}_i$ is also in F . Since F is a face, all non-zero terms of the latter have index $\leq t$. So the negative support is also in $1, 2, \dots, t$. Hence

$$\text{negative support} \subset \{1, \dots, t\} \iff \text{positive support} \subset \{1, \dots, t\}.$$

If the positive and negative support of λ contain indices $> t$ then $f(v_1, \dots, v_t)$ satisfies $\square_\lambda f = 0$ trivially.

Define a new set $\tilde{A} = \{\tilde{\mathbf{a}}_1, \dots, \tilde{\mathbf{a}}_t\} \subset \mathbb{Z}^s$ where $\tilde{\mathbf{a}}_j$ is the projection of \mathbf{a}_j on its first s coordinates. Define a new parameter $\tilde{\beta}$ similarly. The solutions of the form $f(v_1, \dots, v_t)$ of the original GKZ-system satisfy the new GKZ-system corresponding to $H_{\tilde{A}}(\tilde{\beta})$. They all satisfy the additional equations $\partial_i F = 0$ for $i > t$, so they form a proper subspace of the solution space of $H_A(\alpha)$. Hence the system is reducible, contradicting our initial assumption of irreducibility. \square

4 Series solutions

Just as in the classical literature we like to be able to display explicit series solutions for the A-hypergeometric system. In GKZ-theory one chooses $\gamma = (\gamma_1, \dots, \gamma_N)$ such that $\alpha = \gamma_1 \mathbf{a}_1 + \dots + \gamma_N \mathbf{a}_N$ and take as starting point is the formal Laurent series

$$\Phi_{L, \gamma}(v_1, \dots, v_N) = \sum_{\mathbf{l} \in L} \frac{\mathbf{v}^{\mathbf{l} + \gamma}}{\Gamma(\mathbf{l} + \gamma + \mathbf{1})}$$

where we use the short-hand notation

$$\frac{\mathbf{v}^{\mathbf{l} + \gamma}}{\Gamma(\mathbf{l} + \gamma + \mathbf{1})} = \frac{v_1^{l_1 + \gamma_1} \dots v_N^{l_N + \gamma_N}}{\Gamma(l_1 + \gamma_1 + 1) \dots \Gamma(l_N + \gamma_N + 1)}.$$

Note that there is a freedom of choice in γ by shifts over $L \otimes \mathbb{R}$. A priori this series is formal, i.e. there is no convergence. However by making proper

choices for γ we do end up with series that have an open domain of convergence in \mathbb{C}^N .

Choose a subset $\mathcal{I} \subset \{1, 2, \dots, N\}$ with $|\mathcal{I}| = N - r$ such that \mathbf{a}_i with $i \notin \mathcal{I}$ are linearly independent. In [5, Prop 1] we find the following proposition (albeit in a different formulation).

Proposition 4.1 *Define $\pi_{\mathcal{I}} : L \rightarrow \mathbb{Z}^{N-r}$ by $\mathbf{l} \mapsto (l_i)_{i \in \mathcal{I}}$. Then $\pi_{\mathcal{I}}$ is injective and its image is a sublattice of \mathbb{Z}^{N-r} of index $|\det(\mathbf{a}_i)_{i \notin \mathcal{I}}|$.*

We denote $\Delta_{\mathcal{I}} = |\det(\mathbf{a}_i)_{i \notin \mathcal{I}}|$. Choose γ such that $\gamma_i \in \mathbb{Z}$ for $i \in \mathcal{I}$. The formal solution series

$$\Phi = \sum_{\mathbf{l} \in L} \prod_{i \in \mathcal{I}} \frac{v_i^{l_i + \gamma_i}}{\Gamma(l_i + \gamma_i + 1)} \prod_{i \notin \mathcal{I}} \frac{v_i^{l_i + \gamma_i}}{\Gamma(l_i + \gamma_i + 1)}$$

is now a powerseries because the summation runs over the polytope $l_i + \gamma_i \geq 0$ for $i \in \mathcal{I}$ and the other l_j are dependent on $l_i, i \in \mathcal{I}$. Terms where $l_i + \gamma_i < 0$ do not occur because $1/\Gamma(l_i + \gamma_i + 1)$ is zero when $l_i + \gamma_i$ is a negative integer. By slight abuse of language will call the corresponding simplicial cone $l_i \geq 0$ for $i \in \mathcal{I}$ the *sector of summation* with index \mathcal{I} .

Denote the resulting series expansion by $\Phi_{\mathcal{I}, \gamma}$. The following statement, which is a direct consequence of estimates using Stirling's formula for Γ , says that there is a non-trivial region of convergence.

Proposition 4.2 *Let $(\rho_1, \dots, \rho_N) \in \mathbb{R}^N$ be such that $\rho_1 l_1 + \dots + \rho_N l_N > 0$ for all $\mathbf{l} \in L$ with $\forall i \in \mathcal{I} : l_i \geq 0$. Then $\Phi_{\mathcal{I}, \gamma}$ converges for all $\mathbf{v} \in \mathbb{C}^N$ with $|v_i| = t^{\rho_i}$ for sufficiently small $t \in \mathbb{R}_{>0}$.*

A proof can be found for example in [12]. An N -tuple ρ such that $\rho_1 l_1 + \dots + \rho_N l_N > 0$ for all $\mathbf{l} \in L$ with $\forall i \in \mathcal{I} : l_i \geq 0$ will be called a *convergence direction*.

The following statement is a direct Corollary of Proposition 4.1.

Corollary 4.3 *With notations as above, the number of distinct choices modulo L for γ such that $\forall i \in \mathcal{I} : \gamma_i \in \mathbb{Z}$ is $\Delta_{\mathcal{I}}$.*

There is one important assumption we need in order to make this approach work. Namely the guarantee that not too many of the arguments $l_i + \gamma_i$ are a negative integer. Otherwise we might even end up with a power series which is identically zero. The best way to do is to impose the condition $\gamma_i \notin \mathbb{Z}$ for $i \notin \mathcal{I}$. Geometrically, since $\alpha = \sum_{i=1}^N \gamma_i \mathbf{a}_i \equiv \sum_{i \notin \mathcal{I}} \gamma_i \mathbf{a}_i \pmod{\mathbb{Z}^r}$, this condition comes down to the requirement that $\alpha + \mathbb{Z}^r$ does not contain points in a face of the simplicial cone spanned by \mathbf{a}_i with $i \notin \mathcal{I}$. Unfortunately this is stronger than the requirement of non-resonance of $H_A(\alpha)$, as faces of the individual simplicial cones, not necessarily on the boundary of $C(A)$, are involved. However, the condition of non-resonance does turn out to be useful.

Proposition 4.4 *Let \mathcal{I} be as above and suppose the system $H_A(\alpha)$ is non-resonant. Then there exists an open cone C in $L \otimes \mathbb{R}$ such the series $\Phi_{\mathcal{I}, \gamma}$ has non-zero terms for all $\mathbf{l} \in C$.*

Proof. We will use the following observation. The i -th coordinate of $\mathbf{l} \in L$ can be considered as a linear form on L . We shall do so in this proof. Suppose we have a relation $\sum_{i=1}^N \lambda_i l_i = 0$ with $\lambda_i \in \mathbb{R}$. Then there exists a linear form m on \mathbb{R}^r such that $m(\mathbf{a}_i) = \lambda_i$ for $i = 1, \dots, N$.

Denote the set of indices i for which $\gamma_i \notin \mathbb{Z}$ by R . When $|R| = r$ all terms of $\Phi_{\mathcal{I}, \gamma}$ are non-zero and our statement is proven. Suppose $|R| < r$. Then there exist linear relations between the forms l_i with $\lambda_i = 0$ when $i \in R$. Consider the convex hull D of the forms l_i for $i \notin R$. Suppose this hull contains the trivial form $\mathbf{0}$. In other words, there exists a relation with coefficients $\lambda_i \in \mathbb{R}_{\geq 0}$, not all zero, with $\lambda_i = 0$ for all $i \in R$. Hence, by our observation, there exists a non-trivial form m on \mathbb{R}^r such that $m(\mathbf{a}_i) = \lambda_i$ for all i . Hence we have found a non-trivial form with $m(\mathbf{a}_i) \geq 0$ for all i and $m(\mathbf{a}_i) = 0$ for $i \in R$. Therefore the $\mathbb{R}_{\geq 0}$ -span of $\mathbf{a}_i, i \in R$ is contained in a face F of $C(A)$.

Furthermore, $\alpha = \sum_{i=1}^N \gamma_i \mathbf{a}_i \equiv \sum_{i \in R} \gamma_i \mathbf{a}_i \pmod{\mathbb{Z}^r}$. Hence modulo \mathbb{Z}^r the vector α lies in the face F . This contradicts our non-resonance assumption and therefore the convex hull D does not contain $\mathbf{0}$. Consequently, the set of inequalities $l_i \geq 0$, $i \notin R$ has a polyhedral cone with non-empty interior as solution space in \mathbb{R}^{N-r} . The terms in $\Phi_{\mathcal{I}, \gamma}$ with indices inside this cone are non-zero. \square

The following Theorem was one of the discoveries made by Gel'fand, Kapranov and Zelevinsky.

Theorem 4.5 *Let ρ be a convergence direction. Then there exists a regular triangulation T of A such that the summation sectors for which ρ is a convergence direction are given by J^c where J runs through the $(r-1)$ -simplices in T .*

In order to proceed it is now important that different choices of summation sectors give independent series solutions. For this we require the following condition.

Definition 4.6 *For any subset $J \subset \{1, 2, \dots, N\}$ denote $A_J = \{\mathbf{a}_j | j \in J\}$ and let $Q(A_J)$ be the convex hull of the points in A_J .*

Let T be a regular triangulation of A . The parameter α will be called T -nonresonant if $\alpha + \mathbb{Z}^r$ does not contain a point on the boundary of any cone over a $(r-1)$ -simplex $Q(A_J)$ with $J \in T$. We call the system T -resonant otherwise.

Notice that the T -nonresonance condition implies the nonresonance condition. Let us assume that α is T -nonresonant. For any $\mathcal{I} = J^c$ with $J \in T$ and one of the $\text{Vol}(Q(A_J))$ choices of γ we get the series $\Phi_{\mathcal{I}, \gamma}$.

Theorem 4.7 *Under the T -nonresonance condition the power series solutions just constructed form a basis of solutions of $H_A(\alpha)$.*

Proof. To show that the solutions are independent it suffices to show that for any two distinct summation sectors \mathcal{I} and \mathcal{I}' the values of $\gamma_1, \dots, \gamma_N$, as chosen in $\Phi_{\mathcal{I}}$ and $\Phi_{\mathcal{I}'}$, are distinct modulo the lattice L . Suppose they are not distinct modulo L . Then there exists an index $i \in \mathcal{I}'$, but $i \notin \mathcal{I}$ such that $\gamma_i \in \mathbb{Z}$. But this is contradicted by our T -nonresonance assumption.

For every $J \in T$ we get $\text{Vol}(Q(A_J))$ solutions by the different choices of γ . Summing over $J \in T$ shows that we obtain $\sum_{J \in T} \text{Vol}(Q(A_J)) = \text{Vol}(Q(A))$ independent solutions. \square

Given a regular triangulation we can consider the union of all summation domains in L . More precisely, define $\text{supp}(T)$ to be the convex closure of $\cup_{J \in T} \{\mathbf{l} \in L | l_i \geq 0 \text{ for all } i \in J^c\}$. Then $\text{supp}(T)$ will be the common support of all series $\Phi_{\mathcal{I}}$ with $\mathcal{I}^c \in T$. More precisely, denote the set of powerseries in \mathbf{v} with support in $\text{supp}(T)$ by $\mathbb{C}[[\mathbf{v}]]_T$. Note that this set forms a ring by the obvious multiplication. The coefficient ring \mathbb{C} can be extended to the ring of finite linear combinations of powers \mathbf{v}^γ to get the ring denoted by $\mathbb{C}[\mathbf{v}^\gamma][[\mathbf{v}]]_T$. Note that the series constructed above all belong to this ring. In the next section we further extend our coefficient ring to include polynomials in $\log(v_i)$. This larger ring $\mathbb{C}[\log(\mathbf{v}), \mathbf{v}^\gamma][[\mathbf{v}]]_T$ is called a Nilssen ring in [10].

5 T-resonant solutions

In this section we assume that the system is $H_A(\alpha)$ is non-resonant, but not necessarily T -nonresonant. In such a case it is possible to write down a basis of solutions in $\mathbb{C}[\log(\mathbf{v}), \mathbf{v}^\gamma][[\mathbf{v}]]_T$. This is done for example in [10, Ch 3]. We like to reproduce the proof from [10], but in a slightly modified language.

Let T be a regular triangulation of $Q(A)$. This time we assume the system to be T -resonant when we specialise γ to γ^o , say. Let

$$B_{\gamma^o} = \{J \in T \mid \gamma_i^o \in \mathbb{Z} \text{ for all } i \in J^c\}.$$

We say that the simplices $Q(A_J)$ with $J \in B_{\gamma^o}$ are resonating or in resonance with respect to γ^o . In case of T -nonresonance we would get $|B_{\gamma^o}|$ independent series from the specialisation of γ corresponding to $J \in B_{\gamma^o}$. Now we get only one. So we have to find $|B_{\gamma^o}| - 1$ additional series solutions. Just as in the one variable case this will require the use of logarithms of the variables v_i .

Let us denote $b = |B_{\gamma^o}|$. Choose α' such that $H_A(\alpha + \epsilon\alpha')$ is T -nonresonant for every sufficiently small $\epsilon \neq 0$. The b summation sectors J^c with $J \in B_{\gamma^o}$ now give rise to b distinct specialisations of the form $\gamma^o + \epsilon\gamma^{(i)}$ for $i = 1, 2, \dots, b$ producing b independent solutions $\Phi_{\gamma^o + \epsilon\gamma^{(i)}}(\mathbf{v})$ of $H_A(\alpha + \epsilon\alpha')$. Multiply each of these series by $\Gamma(\gamma^o + \epsilon\gamma^{(i)} + \mathbf{1})$ to obtain the solutions

$$\Psi_i(\epsilon, \mathbf{v}) = \sum_{\mathbf{l} \in L} \frac{\Gamma(\gamma^o + \epsilon\gamma^{(i)} + \mathbf{1})}{\Gamma(\mathbf{1} + \gamma^o + \epsilon\gamma^{(i)} + \mathbf{1})} \mathbf{v}^{\mathbf{l} + \gamma^o + \epsilon\gamma^{(i)}}.$$

Note that the coefficients are rational functions of ϵ . Now expand

$$\mathbf{v}^{\mathbf{l} + \gamma^o + \epsilon\gamma^{(i)}} = \sum_{n \geq 0} \frac{\epsilon^n}{n!} (\gamma_1^{(i)} \log v_1 + \dots + \gamma_N^{(i)} \log v_N)^n.$$

Also expand the rational function $\Gamma(\gamma^o + \epsilon\gamma^{(i)} + \mathbf{1})/\Gamma(\mathbf{1} + \gamma^o + \epsilon\gamma^{(i)} + \mathbf{1})$ into a power series in ϵ . We get

$$\Psi_i(\epsilon, \mathbf{v}) = \sum_{n \geq 0} \frac{\epsilon}{n!} \Psi_i^{(n)}(0, \mathbf{v})$$

where $\Psi_i^{(n)}(\epsilon, \mathbf{v})$ denotes the n -th derivative of $\Psi_i(\epsilon, \mathbf{v})$ with respect to ϵ . In particular, $\Psi_i(0, \mathbf{v}) = \Gamma(\gamma^o + \mathbf{1})\Phi_{\gamma^o}(\mathbf{v})$, i.e. all ϵ -series expansions $\Psi_i(\epsilon, \mathbf{v})$ have the same initial term.

Let V_0 be the \mathbb{C} -vector space generated by the $\Psi_i(\epsilon, \mathbf{v})$. Its dimension is b . There is a filtration $V_0 \supset V_1 \supset V_2 \supset \dots$ on V_0 defined by $f(\epsilon, \mathbf{v}) \in V_m$ if f is divisible by ϵ^m . Clearly $\dim(V_M) = 0$ for sufficiently large M . Let $f(\epsilon, \mathbf{v}) \in V_m$. Then $g(\mathbf{v}) = \lim_{\epsilon \rightarrow 0} \epsilon^{-m} f(\epsilon, \mathbf{v})$ is a solution of $H_A(\alpha)$. This is clear for the box-operators \square_1 since they are independent of ϵ . Let Z_i be a homogeneity operator. Then $(Z_i - \alpha_i - \epsilon\alpha'_i)f(\epsilon, \mathbf{v}) = 0$. Divide by ϵ^m and let $\epsilon \rightarrow 0$. Then $(Z_i - \alpha_i)g(\mathbf{v}) = 0$, as desired. Note that $g(\mathbf{v}) \in \mathbb{C}[\log(\mathbf{v}), \mathbf{v}^\gamma][[\mathbf{v}]]_T$.

Let $b_j = \dim(V_j)$ for all j , in particular $b_0 = b$. We choose a basis of V_0 as follows. Take $b_0 - b_1$ elements $f_{b_0}, \dots, f_{b_1+1}$ of V_0 which are linearly independent modulo V_1 . Choose $b_1 - b_2$ elements $f_{b_1}, \dots, f_{b_2+1} \in V_1$ which are independent modulo V_2 , etc. We say that f_i has weight w if $f \in V_w$ and $f \notin V_{w+1}$. Divide f_i by ϵ^w and let $\epsilon \rightarrow 0$. Denote the limit by $g_i(\mathbf{v})$. By construction elements $g_i(\mathbf{v})$ coming from f_i of the same weight are linearly independent. Elements $g_i(\mathbf{v})$ coming from f_i with distinct weights are independent because the series expansion have different degrees in the $\log(v_i)$. Hence the series $g_i(\mathbf{v})$ provide the desired b independent solutions of $H_A(\alpha)$. Thus we obtained the Theorem of Saito-Sturmfels-Takayama [10, Thm 3.5.1] for the case of non-resonant systems (in their book the author also produce bases of resonant systems).

Theorem 5.1 (Saito-Sturmfels-Takayama) *Suppose $H_A(\alpha)$ is non-resonant. For any regular triangulation of $Q(A)$ there exists a space of solutions to $H_A(\alpha)$ in the ring $\mathbb{C}[\log(\mathbf{v}), \mathbf{v}^\gamma][[\mathbf{v}]]_T$ of \mathbb{C} -dimension $\text{Vol}(A)$.*

By a Theorem of Adolphson [1, Corollary 5.20] the rank of $H_A(\alpha)$ equals $\text{Vol}(A)$ when the system is non-resonant. Hence we get the following.

Corollary 5.2 *When $H_A(\alpha)$ is non-resonant the system of solutions in Theorem 5.1 provides a basis of solutions to $H_A(\alpha)$ in $\mathbb{C}[\log(\mathbf{v}), \mathbf{v}^\gamma][[\mathbf{v}]]_T$.*

6 Non-resonant systems

In this section we prove Theorem 1.2. Suppose we have a non-resonant system and an operator $P \in K[\partial]$ which annihilates a non-trivial solution f in the solution space of $H_A(\alpha)$.

First we show the existence of such an f which is of the form a power series of the type Φ_γ , as in the previous two sections. Fix a convergence direction ρ_1, \dots, ρ_N and let T be the corresponding regular triangulation of $Q(A)$.

Corollary 5.2 provides a basis of solutions in $\mathbb{C}[\log(\mathbf{v}), \mathbf{v}^\gamma][[\mathbf{v}]]_T$. Consider these solutions as analytic functions on an open neighbourhood of the set V given by $|v_1| = t^{\rho_1}, \dots, |v_N| = t^{\rho_N}$ for t sufficiently small. The fundamental group $\pi_1(V)$ is generated by $v_j = t^{\rho_j} e^{2\pi i x}$, $x \in [0, 1]$ for any j and v_i fixed for all $i \neq j$. The corresponding monodromy group is an abelian group and so is its restriction to the common solution space of $H_A(\alpha)$ and $P(f) = 0$. Since the monodromy group is abelian, there exists a one-dimensional invariant subspace. The character, with which $\pi_1(V)$ acts on this space, uniquely determines a solution of the form Φ_γ .

In the terminology of [9, Thm 2.7] the solution Φ_γ is a fully supported solution by virtue of Proposition 4.4. Theorem 2.7 of [9] implies that the operator P lies in $\mathcal{H}_A(\alpha)$. Hence we conclude that $H_A(\alpha)$ is irreducible. \square

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